Final Exam Phys 3356 Spr 2005

1.1

$$L = T - V = \boxed{\tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2}$$

$$p \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = p\dot{x} - L = m\dot{x}^2 - \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} k x^2$$

$$= \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} k x^2 = \boxed{\frac{p^2}{2m} + \tfrac{1}{2} k x^2}$$

1.2

$$\dot{p} = -\frac{\partial H}{\partial x} = \boxed{-kx} \qquad \dot{x} = \frac{\partial H}{\partial p} = \boxed{\frac{p}{m}}$$

1.3

$$\ddot{x} = \frac{\dot{p}}{m} = \boxed{-\frac{k}{m} x}$$

2.1

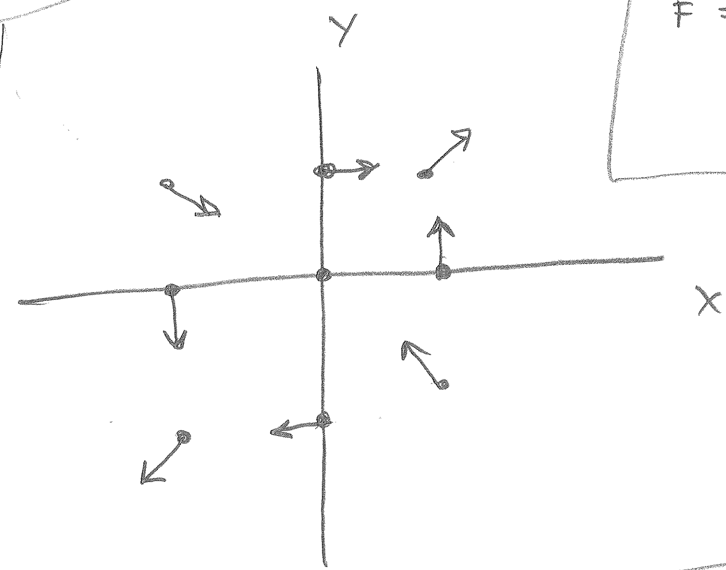
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \cdots$$

$$= \hat{x}\Big(-\big[\ln x\,(-\sin y)\cos z\big] - (-1)\big[\ln x \sin y\,(+\cos z)\big]\Big)$$

$$+ \cdots \neq 0$$ $\boxed{\text{No}}$ not conservative

2.2

$\vec{F} = -\vec{\nabla} V = y\,\hat{x} + x\,\hat{y}$



3.

$x = (x_1, x_2, x_3, ict)$ $\qquad x \cdot x = x_1^2 + x_2^2 + x_3^2 - c^2t^2$

$x = (x_1, x_2, x_3, ct)$

$$x \cdot x \equiv (x_1\ x_2\ x_3\ ct)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ ct \end{pmatrix}$$

$x_\mu = (x_0, x_1, x_2, x_3) \equiv (-x^0, x^1, x^2, x^3)$

$x \cdot x \equiv x_\mu x^\mu = \sum_{\mu=0}^{3} x_\mu x^\mu \qquad x_0 = ct$

4

$$T = \int dt = 4\int_0^A \frac{dx}{\dot{x}(x, A)}$$

$$\dot{x}^2 = \frac{A^4}{2} - \frac{x^4}{2} \Rightarrow \dot{x} = \pm\sqrt{\tfrac{1}{2}(A^4 - x^4)}$$

$$\Rightarrow T = \int dt = 4\int_0^A \frac{dx}{\sqrt{\frac{1}{2}A^4 - \frac{1}{2}x^4}} = 4\sqrt{2}\int_0^A \frac{dx}{\sqrt{A^4 - x^4}}$$

change of variables

$$y = \frac{x}{A} \qquad \begin{matrix} x = 0 & y = 0 \\ x = A & y = 1 \end{matrix} \qquad \begin{matrix} dx = A\,dy \\ x = Ay \end{matrix}$$

$$\Rightarrow T = 4\sqrt{2}\int_0^1 \frac{A\,dy}{\sqrt{A^4 - A^4y^4}} = 4\sqrt{2}\,\frac{\cancel{A}}{A^{\cancel{2}}}\int_0^1 \frac{dy}{\sqrt{1-y^4}}$$

$$\Rightarrow \boxed{\omega = \frac{2\pi}{T} = \frac{\cancel{2}\pi A}{\cancel{4}_{2}\sqrt{2}\int_0^1 \frac{dy}{\sqrt{1-y^4}}} \Rightarrow \omega(A) = \left(\frac{\pi}{\sqrt{2}\int_0^1 \frac{dy}{\sqrt{1-y^4}}}\right)A}$$

$\int_0^1 \frac{dy}{\sqrt{1-y^4}}$ is just a number

5

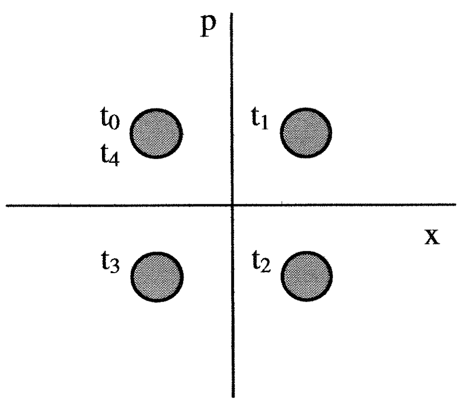
Consider the following three simple harmonic oscillators (aka, springs):

(1) an undriven, undamped simple harmonic oscillator
(2) a damped simple harmonic oscillator
(3) a driven simple harmonic oscillator

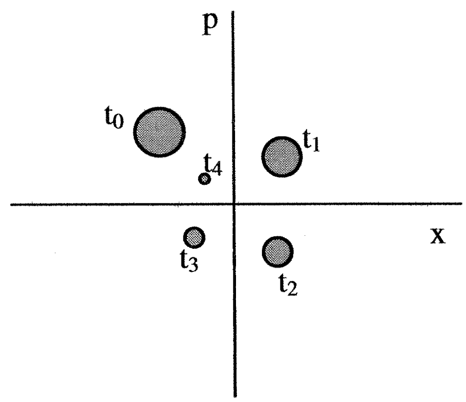
5.1 (a) State whether each of the three systems is a Hamiltonian canonical system. (Remember, in order to be a Hamiltonian canonical system, its Hamiltonian function must exist and accurately and fully portray the system).

> ***(1) Hamiltonian canonical system***
> ***(2) not a Hamiltonian canonical system*** if it's linear damping
> ***(3) ~~not a Hamiltonian canonical system~~*** May or May Not

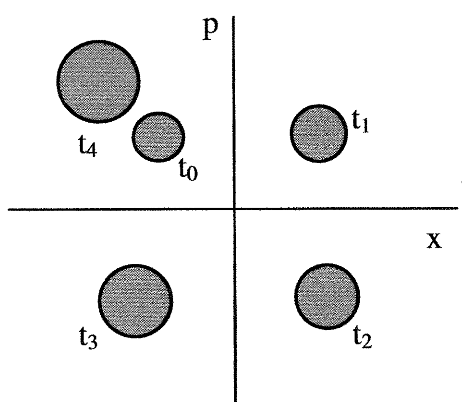
5.2 (b) Given that each of the simple harmonic oscillators has a period of 2π and given an ensemble of systems at $t_0 = 0$, on each of the following three phase space plots, sketch and label the ensemble of systems at time $t_1 = \pi/2$, time $t_2 = \pi$, time $t_3 = 3\pi/2$, and time $t_4 = 2\pi$.



(1) undriven, undamped



(2) damped



Could be like this

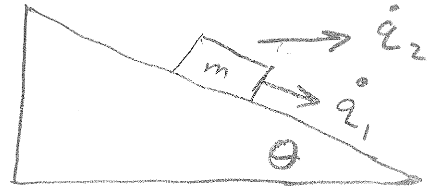
(3) driven

5.3 (c) For which of the three systems is the area of the ensemble conserved over time? Why? (hint: check the correlation between the systems whose areas are conserved and your answers to part a)

> ***The area of the ensemble is conserved over time for system (1) because system (1) is a Hamiltonian canonical system. (Therefore, Liouville's Theorem applies, but the student does not need to mention Liouville's Theorem for full credit.)***

Must is time independent? Too ?

6.1 $L = T - V$



$$T_m = \frac{1}{2} m \left(V_x^2 + V_y^2\right)$$

$$V_x = \dot{q}_2 + \dot{q}_1 \cos\theta \qquad V_y = \dot{q}_1 \sin\theta$$

$$V_x^2 = \dot{q}_2^2 + 2\dot{q}_2\dot{q}_1\cos\theta + \dot{q}_1^2\cos^2\theta$$

$$V_y^2 = \dot{q}_1^2 \sin^2\theta$$

$$V_x^2 + V_y^2 = \dot{q}_2^2 + \dot{q}_1^2 + 2\dot{q}_2\dot{q}_1\cos\theta$$

$$L = T_m + T_M - V_m$$

$$= \frac{1}{2} m \dot{q}_2^2 + \frac{1}{2} m \dot{q}_1^2 + m\dot{q}_1\dot{q}_2\cos\theta + \frac{1}{2} M \dot{q}_2^2$$

$$- \left(- m g q_1 \sin\theta\right)$$

$$\Rightarrow \boxed{L = \frac{1}{2}(M+m)\dot{q}_2^2 + \frac{1}{2} m \dot{q}_1^2 + m\dot{q}_1\dot{q}_2\cos\theta + m g q_1 \sin\theta}$$

6.2

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial q_1} \Rightarrow m\ddot{q}_1 + \frac{d}{dt}\left[m\dot{q}_2\cos\theta\right] = m g \sin\theta$$

$$\Rightarrow \boxed{\cancel{m}\ddot{q}_1 + \cancel{m}\ddot{q}_2\cos\theta = \cancel{m} g \sin\theta \quad (1)}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = \frac{\partial L}{\partial q_2} \Rightarrow \boxed{(M+m)\ddot{q}_2 + m\ddot{q}_1\cos\theta = 0 \quad (2)}$$

6.2 (continued)

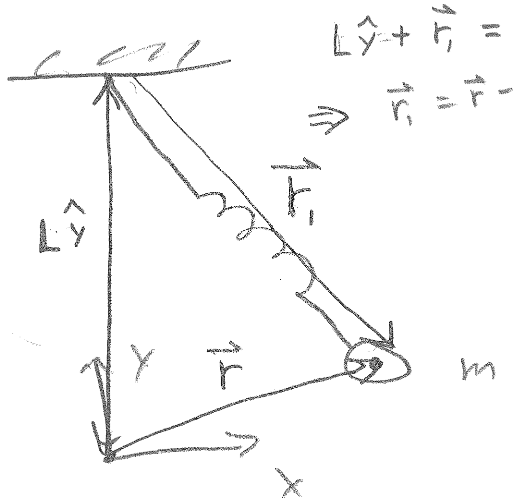
$$\text{(2)} \Rightarrow \quad \ddot{q}_2 = -\frac{m}{M+m}\,\ddot{q}_1 \cos\theta$$

$\Downarrow$

$$\text{(1)} \Rightarrow \quad \cancel{m}\ddot{q}_1 - \frac{m^{\cancel{2}}}{M+m}\cos^2\theta\;\ddot{q}_1 = \cancel{m}g\sin\theta$$

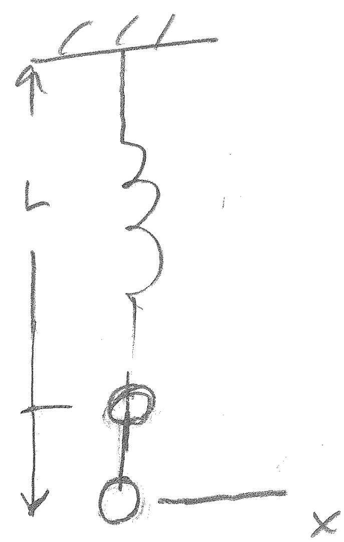
$$\Rightarrow \boxed{\ddot{q}_1 = g\sin\theta\,\frac{1}{1-\frac{m}{M+m}\cos^2\theta}}$$

7



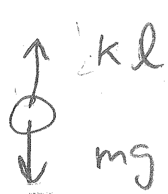
$$L\hat{y} + \vec{r}_1 = \vec{r}$$

$$\Rightarrow \vec{r}_1 = \vec{r} - L\hat{y}$$



$$L = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 - V_g - V_k$$

$$V_g = mgy \qquad V_k = \tfrac{1}{2}k(\Delta l)^2$$



$$\Sigma F_y = 0 \Rightarrow kl = mg \Rightarrow l = \frac{mg}{k}$$

$$V_k = \tfrac{1}{2}k\left(\left(L-l\right) - |\vec{r}_1|\right)^2$$

7 (continued)

$$V_k = \frac{1}{2}k\left[(L-\ell) - |\vec{r} - L\hat{y}| \right]^2$$

$$= \frac{1}{2}k\left[(L-\ell) - \left(x^2 + (y-L)^2 \right)^{1/2} \right]^2$$

$$= \frac{1}{2}k\left[(L-\ell) - L\left(\left(\frac{x}{L}\right)^2 + \left(1-\frac{y}{L}\right)^2 \right)^{1/2} \right]^2$$

$$= \frac{1}{2}k\left\{ (L-\ell) - L\Big[1 \underbrace{- 2\frac{y}{L} + \left(\frac{y}{L}\right)^2 + \left(\frac{x}{L}\right)^2}_{\text{small}} \Big]^{1/2} \right\}^2$$

$$(1+\epsilon)^p \simeq 1 + p\epsilon + \frac{p(p-1)}{2}\epsilon^2 + \cdots$$

$$\epsilon \equiv -2\frac{y}{L} + \left(\frac{y}{L}\right)^2 + \left(\frac{x}{L}\right)^2$$

$$(1+\epsilon)^{1/2} \simeq 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \cdots$$

$$\Rightarrow V_k \simeq \frac{1}{2}k\left\{ (L-\ell) - L\left\{ 1 - \frac{y}{L} + \frac{1}{2}\left(\frac{y}{L}\right)^2 + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{8}\left[4\frac{y^2}{L^2} + \mathcal{O}\left(\left(\frac{y}{L}\right)^3\right) \right] \right\} \right\}^2$$

where $\mathcal{O}\left(\left(\frac{y}{L}\right)^3\right)$ is small like $\left(\frac{y}{L}\right)^3$ or $\left(\frac{x^2 y}{L^3}\right)$

$$\Rightarrow V_k \simeq \frac{1}{2}k\left\{ \cancel{L} - \ell - \cancel{L} + y - \cancel{\frac{1}{2}\frac{y^2}{L}} - \frac{1}{2}\frac{x^2}{L} - \cancel{\frac{1}{2}\frac{y^2}{L}} \right\}^2$$

$$\simeq \frac{1}{2}k\ell^2 + \frac{1}{2}ky^2 - \frac{1}{2}k\,2\ell y + \frac{1}{2}k\frac{\ell}{L}x^2$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \cancel{mgy} - \frac{1}{2}k\ell^2 - \frac{1}{2}ky^2 + \cancel{k\ell y} - \frac{1}{2}k\ell\frac{x^2}{L}$$

$$\ell = \frac{mg}{k} \Rightarrow L = \underbrace{\frac{1}{2}m\dot{x}^2 - \frac{1}{2}\frac{mg}{L}x^2}_{x\text{-mode}} + \underbrace{\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2}_{y\text{-mode}} + \text{const}$$

7 (continued)

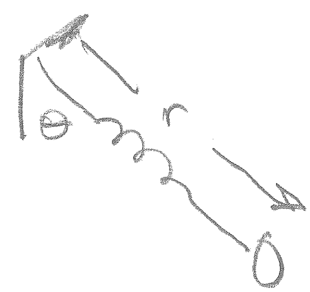
$$\Rightarrow \quad \ddot{x} = -\frac{g}{L}x \qquad \ddot{y} = -\frac{k}{m}y$$

x and y are the modal degrees of freedom

$$\boxed{\omega_1 = \sqrt{\frac{g}{L}} \qquad \vec{a}_1 = \begin{pmatrix}1\\0\end{pmatrix} \qquad \text{X - mode}} \qquad \boxed{\omega_2 = \frac{k}{m} \qquad \vec{a}_2 = \begin{pmatrix}0\\1\end{pmatrix} \qquad \text{y - mode}}$$

7 <u>Other way</u>

from



<u>from Exam 2 Prob 2 Phy 3355</u>

① $m\ddot{r} = mr\dot{\theta}^2 + k(\ell - r) + mg\cos\theta$

② $\ddot{\theta} = -2\frac{\dot{r}\dot{\theta}}{r} - \frac{g}{r}\sin\theta$

change of variables

$$r\sin\theta = X \qquad r\cos\theta = L - Y$$

$$\theta = \frac{x}{r}$$

small $x, \theta, y \Rightarrow r\theta = X \qquad r = L - Y$

y-mode

① $\Rightarrow \boxed{-\ddot{y} = 0 + ky} + 0 + 0 \qquad \ddot{\theta} = 0 - \frac{g\theta}{r}$

7 (continued)

$$\ddot{\theta} = -\frac{g\theta}{r} \quad \text{and} \quad \overbrace{(L-y)}^{r}\,\theta = x$$

$$\Rightarrow \theta = \frac{x}{(L-y)} = \frac{x}{L(1-\frac{y}{L})} \simeq \frac{x}{L}$$

$$\Rightarrow \ddot{\theta} \simeq \ddot{x}L \Rightarrow \ddot{x} = -\frac{1}{L}\frac{g\theta}{r} \simeq -\frac{1}{L}\, g\, \frac{x/\cancel{L}}{\cancel{L}}$$

$$\Rightarrow \boxed{\ddot{x} \simeq -\frac{g}{L}x}$$

x - mode

8.

by ispektion this is the second mode

$\omega_2^2 = \frac{2k}{m}$

$$\boxed{\vec{q}(t) = \begin{pmatrix} -b \cos\omega_2 t \\ 0 \\ b \cos\omega_2 t \end{pmatrix}}$$

$N=2$
$n=3$

$\omega_2 = 2\omega_0 \sin\left(\frac{2\pi}{6+2}\right) = 2\omega_0 \sin\left(\frac{\pi}{4}\right)$

$= 2\omega_0\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}\,\omega_0 = \sqrt{2}\sqrt{\frac{k}{m}}$

9.1

$$\boxed{L = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m \underbrace{R^2 \sin^2\theta}\, \omega^2 - mgR(1-\cos\theta)}$$

9.2

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \Rightarrow \cancel{m}R^2\ddot{\theta} = \cancel{m}R^2\omega^2 \sin\theta\cos\theta - \cancel{m}gR\sin\theta$

$\Rightarrow \ddot{\theta} = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta$

$\omega^2 < \frac{g}{R} \quad \theta \approx 0 \Rightarrow \ddot{\theta} = -\left(\frac{g}{R} - \omega^2\right)\theta$

$$\boxed{\text{ang. freq} = \sqrt{\frac{g}{R} - \omega^2}}$$

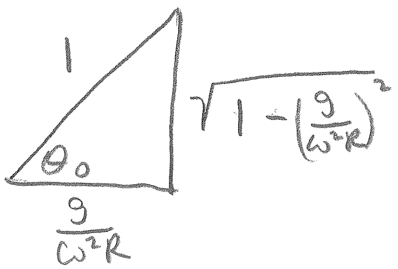
9.3 $\cos\theta_0 = \frac{g}{\omega^2 R}$

$\overbrace{\frac{d}{d\theta}\left(\omega^2\cos\theta\sin\theta - \frac{g}{R}\sin\theta\right)\Big|_{\theta=\theta_0}}^{-(\text{ang freq})^2}$

$= \left[\omega^2(-\sin^2\theta) + \omega^2\cos^2\theta - \frac{g}{R}\cos\theta\right]\Big|_{\theta=\theta_0}$

9.2 (continued)

$$= \omega^2\left[-\sin^2\left(\cos^{-1}\left(\frac{g}{\omega^2 R}\right)\right)\right] + \cancel{\omega^2\left(\frac{g}{\omega^2 R}\right)^2} - \cancel{\frac{g}{R}\,\frac{g}{\omega^2 R}}$$



$$\sin\theta_0 = \sqrt{1-\left(\frac{g}{\omega^2 R}\right)^2}$$

$$\Rightarrow (\text{ang freq})^2 = \omega^2\left[1-\left(\frac{g}{\omega^2 R}\right)^2\right]$$

$$= \omega^2 - \left(\frac{g}{\omega R}\right)^2$$

$$\Rightarrow \boxed{\text{ang freq.} = \sqrt{\omega^2 - \left(\frac{g}{\omega R}\right)^2}}$$