1. Put your name at the top of every paper you hand in.
2. Write the problem number and section of the problem on your work.
3. Box all your answers.
4. All that you hand in must be your own work (Honor Code).

1 [12 points] Hamiltonian - Simple Harmonic Oscillator

A 1-D simple harmonic oscillator has mass m, and spring constant k. There is no damping. Let x be the position of the oscillator.

1.1 (5) Write the Lagrangian and the Hamiltonian

Write the Lagrangian $L(\dot{x}, x)$ and then the Hamiltonian $H(p, x)$ for this 1-D simple harmonic oscillator.

1.2 (4) Write Hamilton's equations of motion

Find Hamilton's equations of motion, that is $\dot{p}$ and $\dot{x}$ as a function of m, k, the momentum p, and the position x.

1.3 (3) Compare to $\ddot{x}$ equation of motion

Using your results from section 1.2, find $\ddot{x}$ as a function of m, k, the position x, and velocity $\dot{x}$.

2 [10 points] Force fields

2.1 (5) Conservative force ?

Is the force field $\vec{F}(x, y, z) = \frac{\cos y \, \sin z}{x} \, \hat{x} \; - \ln x \, \sin y \, \sin z \, \hat{y} \; - \ln x \, \cos y \, \cos z \, \hat{z}$ conservative?

2.2 (5) Drawing force fields

Sketch the force field (draw force vectors at numerous representative points) for the following scalar potential energy: $V(x, y) = -xy$.

3 [6 points] Special relativity

In studying special relativity we where introduced to many different four-vectors whose corresponding four-scalar was invariant under a Lorentz transformation. Show three different mathematical notations to represent the four-position four-vector and it's corresponding four-scalar. For example one form could be $x = (x_1, x_2, x_3, ?)$ and $x \cdot x = ?$, where each "?" must be filled in with different expressions.

4 [10 points] Nonlinear dynamics

A undriven and undamped highly nonlinear oscillator has the equation of motion $\ddot{x} = -x^3$. Find an expression for the angular frequency of oscillation of this oscillator as a function of the amplitude of the oscillator, which is defined by

$$\frac{A^4}{4} \equiv \frac{x^4}{4} + \frac{\dot{x}^2}{2}. \tag{4.1}$$

Simplify the A dependence in your expression. Hints: *Start with the energy equation. Find the period as* $\int dt = 4 \int_0^A \frac{dx}{\dot{x}(x, A)}$. *Factor* A *outside of the integral.*

5 [14 points] Hamiltonian systems

Consider the following three simple harmonic oscillators:

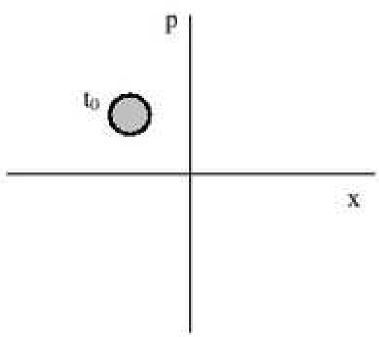
1. an undriven, undamped simple harmonic oscillator
2. a damped simple harmonic oscillator (no driving)
3. a driven simple harmonic oscillator (no damping)

5.1 (5) Hamiltonian system or not

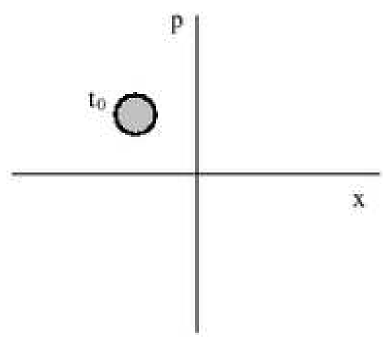
State whether each of the three systems is a Hamiltonian canonical system or not. (Remember, in order to be a Hamiltonian canonical system, its Hamiltonian function must exist and accurately and fully portray the system).

5.2 (5) Ensemble

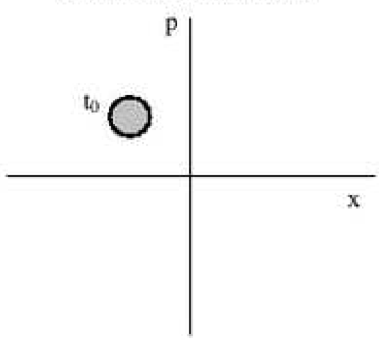
Given that each of the simple harmonic oscillators has a period of 2π and given an ensemble of systems at $t_0 = 0$, on each of the following three phase space plots, sketch and label the ensemble of systems at time $t_1 = \frac{\pi}{2}$, time $t_2 = \pi$, time $t_3 = \frac{3\pi}{4}$, and time $t_4 = 2\pi$. Start by coping these figures onto your solutions paper.



(1) undriven, undamped



(2) damped

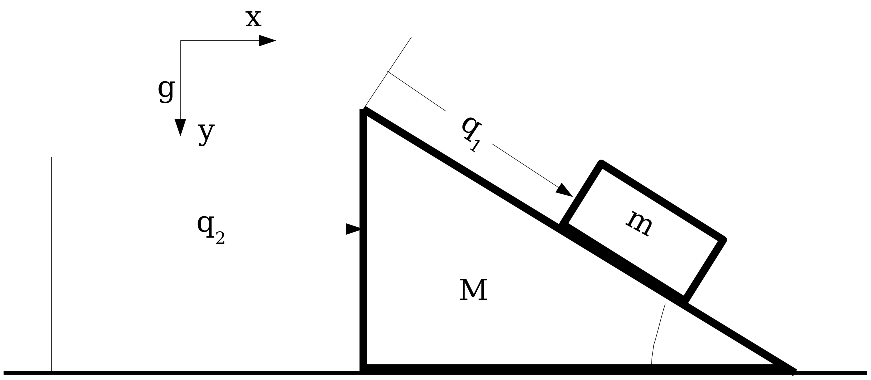


(3) driven

5.3 (4) Phase space area conserved over time?

For which of the three systems is the area of the ensemble conserved over time? Why? (hint: check the correlation between the systems whose areas are conserved and your answers to part 5.1)

6 [15 points] Lagrangian - Sliding block and wedge



A block, with mass m, slides without friction on an incline plane with pitch angle θ measured from the horizontal. The incline plane is a wedge shaped block, with mass M, that slides without friction on the horizontal floor. Gravity acts on both blocks. As the block, with mass m, slides down the plane the wedge slides in the negative x direction.

6.1 (6) Lagrangian

Write the Lagrangian $L(\dot{q}_1, \dot{q}_2, q_1, q_2, t)$ for this system.

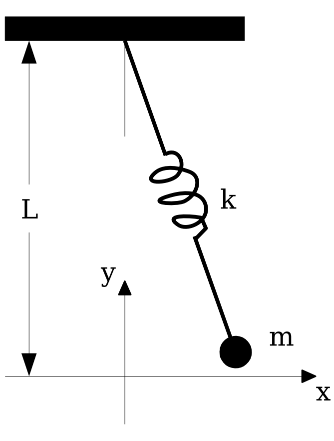
6.2 (5) $\ddot{q}_1$ and $\ddot{q}_2$

Find the equations of motion for q_1 and q_2.

6.3 (4) $\ddot{q}_1$

Find $\ddot{q}_1$ in terms of m, M, θ, and g (without $\ddot{q}_2$).

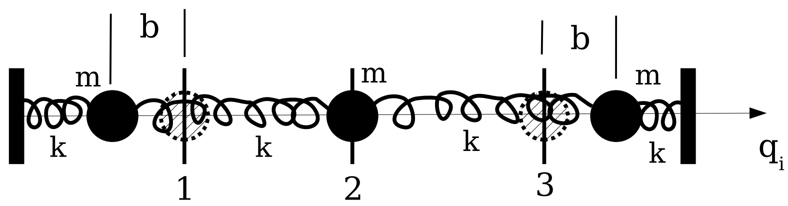
7 [10 points] Normal Modes



The mass of this simple springy pendulum is m. This pendulum can swing and spring up and down. The rest equilibrium position of the pendulum mass is at $x = 0$ and $y = 0$. The rest equilibrium length of the pendulum is L. The mass moves in the x-y plane.

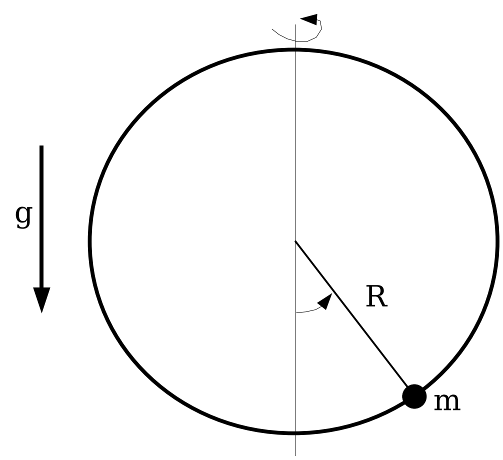
Find the normal angular frequencies and eigen-vectors in terms of g, k, m, L, and use x and y as your coordinate system.

8 [8 points] Linear array of coupled harmonic oscillators



All four springs have spring constant k. All the three masses have mass m. The two outer masses are pulled to the outside a distance b and released from rest at the same time, $t = 0$. What is the position as a function of time of all three masses, that's $q_1(t)$, $q_2(t)$, and $q_3(t)$. Hint: *See Fowles section 11.5.*

9 [15 points] Small oscillations



A bead of mass m slides without fiction on a hoop of radius R. The hoop rotates with a constant angular speed ω about a vertical axis that goes through the center of the hoop. Let θ be the angular position of the bead on the hoop, with the lowest bead position being at $\theta = 0$.

9.1 (6) Lagrangian

Find the exact Lagrangian, $L(\dot{\theta}, \theta)$, for this system.

9.2 (5) Frequency of oscillation about $\theta = 0$

Find the angular frequency of small oscillations about $\theta = 0$, assuming that $\omega^2 < \frac{g}{R}$.

9.3 (4) Frequency of oscillation about $\theta \neq 0$ with larger ω

Find the angular frequency of small oscillations about $\theta \neq 0$, assuming that $\omega^2 > \frac{g}{R}$.