Hamiltonian 6

1. $L = T - V = \frac{1}{2} m \dot{l}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$

$= \frac{1}{2} m \alpha^2 + \frac{1}{2} m (l_0 - \alpha t)^2 \dot{\theta}^2 + mgl\cos\theta$

$$\Rightarrow \boxed{L = \frac{1}{2} m \alpha^2 + \frac{1}{2} m (l_0 - \alpha t)^2 \dot{\theta}^2 + mg(l_0 - \alpha t)\cos\theta}$$

2. $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt}\left[m (l_0 - \alpha t)^2 \dot{\theta} \right] = -mg(l_0 - \alpha t)\sin\theta$

$\Rightarrow m (l_0 - \alpha t)^2 \ddot{\theta} + 2m (l_0 - \alpha t)(-\alpha)\dot{\theta} = -mg(l_0 - \alpha t)\sin\theta$

$$\Rightarrow \boxed{(l_0 - \alpha t)\ddot{\theta} = +2\alpha\dot{\theta} - g\sin\theta}$$

3. $$\boxed{p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m (l_0 - \alpha t)^2 \dot{\theta}}$$

4. $h = p_\theta \dot{\theta} - L = m (l_0 - \alpha t)^2 \dot{\theta}^2 - \frac{1}{2} m \alpha^2 - \frac{1}{2}(l_0 - \alpha t)^2 \dot{\theta}^2 - mg(l_0 - \alpha t)\cos\theta$

$= \frac{1}{2} m (l_0 - \alpha t)^2 \dot{\theta}^2 - \frac{1}{2} m \alpha^2 - mg(l_0 - \alpha t)\cos\theta$

$$\Rightarrow \boxed{H = \frac{p_\theta^2}{2m(l_0 - \alpha t)^2} - \frac{1}{2} m \alpha^2 - mg(l_0 - \alpha t)\cos\theta}$$

5. $$\boxed{\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mg(l_0 - \alpha t)\sin\theta}$$

$$\boxed{\dot{\theta} = -\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m(l_0 - \alpha t)^2}}$$

Checking 5. with 2.

$$\ddot{\theta} = \frac{\dot{P}_\theta}{m(l_0-\alpha t)^2} + \frac{2P_\theta(-\alpha)}{m(l_0-\alpha t)^3}$$

$$\Rightarrow \ddot{\theta} = -\frac{mg(l_0-\alpha t)\sin\theta}{m(l_0-\alpha t)^2} + \frac{2m(l_0-\alpha t)^2\dot{\theta}\alpha}{m(l_0-\alpha t)^3}$$

$$\Rightarrow (l_0-\alpha t)\ddot{\theta} = +2\alpha\dot{\theta} - g\sin\theta$$ same as in 2.

6. $$\boxed{E-H} = \frac{P_\theta^2}{2m(l_0-\alpha t)^2} + \frac{1}{2}m\alpha^2 - mg(l_0-\alpha t)\cos\theta$$
$$- \frac{P_\theta^2}{2m(l_0-\alpha t)^2} + \frac{1}{2}m\alpha^2 + mg(l_0-\alpha t)\cos\theta$$

$$= \boxed{m\alpha^2}$$