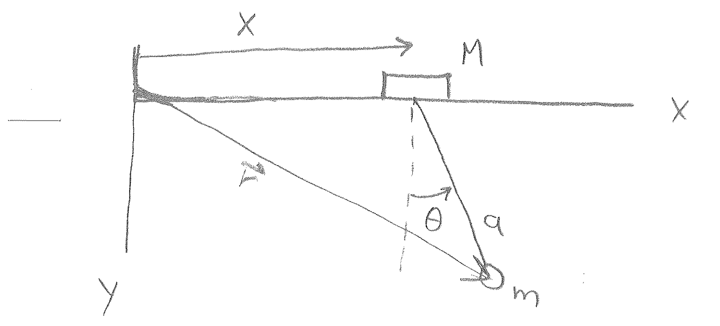
11.20



$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \dot{r}^2$$

$$\vec{r} = x\hat{x} + a\sin\theta\,\hat{x} + a\cos\theta\,\hat{y}$$

$$\dot{\vec{r}} = (\dot{x} + a\dot{\theta}\cos\theta)\hat{x} - a\dot{\theta}\sin\theta\,\hat{y}$$

$$\dot{r}^2 = \dot{x}^2 + 2a\dot{x}\dot{\theta}\cos\theta + a^2\dot{\theta}^2\cos^2\theta + a^2\dot{\theta}^2\sin^2\theta$$

$$= \dot{x}^2 + 2a\dot{x}\dot{\theta}(\underbrace{\cancel{\cos\theta}}_{1}) + a^2\dot{\theta}^2 \qquad \theta << 1$$

$$\Rightarrow T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \dot{x}^2 + m a \dot{x}\dot{\theta} + \frac{1}{2} m a^2 \dot{\theta}^2$$

$$V = mga(1 - \cos\theta) \simeq \frac{1}{2} mga\theta^2 = \frac{1}{2}\frac{mg}{a}(a\theta)^2$$

$$\Rightarrow \quad T = \frac{1}{2}(\dot{x} \;\; a\dot{\theta}) \underbrace{\begin{pmatrix} (M+m) & m \\ m & m \end{pmatrix}}_{\overleftrightarrow{M}} \begin{pmatrix} \dot{x} \\ a\dot{\theta} \end{pmatrix}$$

$$V = \frac{1}{2}(x \;\; a\theta) \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \frac{mg}{a} \end{pmatrix}}_{\overleftrightarrow{K}} \underbrace{\begin{pmatrix} x \\ a\theta \end{pmatrix}}_{\vec{q}} \quad \Rightarrow$$

$$\overleftrightarrow{K}\vec{q} + \overleftrightarrow{M}\ddot{\vec{q}} = 0$$

plug $x = a_1 \cos(\omega t - \delta)$
$a\theta = a_2 \cos(\omega t - \delta)$

$$\Rightarrow \overleftrightarrow{K}\vec{a} - \omega^2 \overleftrightarrow{M}\vec{a} = 0 \Rightarrow (\overleftrightarrow{K} - \omega^2\overleftrightarrow{M})\vec{a} = 0$$

$$\Rightarrow \det\left|\overleftrightarrow{K} - \omega^2\overleftrightarrow{M}\right| = 0$$

$$\Rightarrow \det\left|\begin{pmatrix} 0 & 0 \\ 0 & \frac{mg}{a} \end{pmatrix} - \omega^2 \begin{pmatrix} M+m & m \\ m & m \end{pmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix} -\omega^2(M+m) & -\omega^2 m \\ -\omega^2 m & \frac{mg}{a} - \omega^2 m \end{vmatrix} = 0$$

$$\Rightarrow -\omega^2(M+m)\left(\frac{mg}{a} - \omega^2 m\right) - (\omega^2)^2 m^2 = 0$$

$$\Rightarrow (\omega^2)^2\left[Mm + \cancel{m^2} - \cancel{m^2}\right] - \omega^2(M+m)\frac{mg}{a} = 0$$

$$\Rightarrow \omega^2\left[\omega^2 Mm - (M+m)\frac{mg}{a}\right] = 0$$

$$\Rightarrow \omega_1^2 = 0 \quad \text{and} \quad \omega_2^2 = \frac{(M+m)}{M}\frac{g}{a}$$

ω_1 is a mode that does not oscillate

$$\left(\overleftrightarrow{K} - \omega^2\overleftrightarrow{M}\right)\vec{a} = 0 \Rightarrow \begin{aligned} -\omega^2(M+m)a_1 - \omega^2 m\, a_2 &= 0 \\ -\omega^2 m\, a_1 + \left(\frac{mg}{a} - \omega^2 m\right)a_2 &= 0 \end{aligned}$$

for $\omega_1 = 0 \Rightarrow a_2 = 0$ $\quad a_1$ is not determined

the equations of motion are

$$\overleftrightarrow{M}\ddot{\vec{q}} + \overleftrightarrow{K}\vec{q} = 0 \Rightarrow \left.\begin{aligned} (M+m)\ddot{x} + ma\underset{0}{\cancel{\ddot{\theta}}} &= 0 \quad (1) \\ m\ddot{x} + ma\underset{0}{\cancel{\ddot{\theta}}} + \frac{mg}{a}(a\underset{0}{\cancel{\theta}}) &= 0 \end{aligned}\right\} \Rightarrow \ddot{X} = 0$$

$\theta = 0$

$$(1) \Rightarrow (M+m)\ddot{x} = 0 \Rightarrow \dot{x} = V_{x0} \Rightarrow X(t) = V_{x0}t + X_0$$

where V_{x0} and X_0 are constants of integration

so
$$\boxed{\omega_1 = 0 \qquad \text{mode } 1 \quad \vec{Q}_1 = \begin{pmatrix} V_{x0}t + X_0 \\ 0 \end{pmatrix}}$$

mode 2 $\qquad \omega_2^2 = \frac{M+m}{M}\frac{g}{a}$

$$\left(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M}\right)\vec{a} = 0 \Rightarrow -\omega^2 m a_1 + \left(\frac{mg}{a} - \omega^2 m\right) a_2 = 0$$

$$\Rightarrow \frac{a_2}{a_1} = \frac{m\omega^2}{\frac{mg}{a} - \omega^2 m} = \frac{m\frac{M+m}{M}\frac{g}{a}}{\frac{mg}{a} - \frac{M+m}{M}\frac{g}{a}m} = \frac{\frac{mg}{a}\left(\frac{M+m}{M}\right)}{\frac{mg}{a}\left(1 - \frac{M+m}{M}\right)}$$

$$= \frac{M+m}{\cancel{M}\left(\frac{\cancel{M}-\cancel{M}-m}{\cancel{M}}\right)} = -\frac{M+m}{m}$$

$$\Rightarrow \boxed{\vec{Q}_2 = \begin{pmatrix} 1 \\ -\frac{M+m}{m} \end{pmatrix} \cos\left(\sqrt{\frac{M+m}{M}\frac{g}{a}}\, t - \delta\right)}$$

mode 2

Not normalized