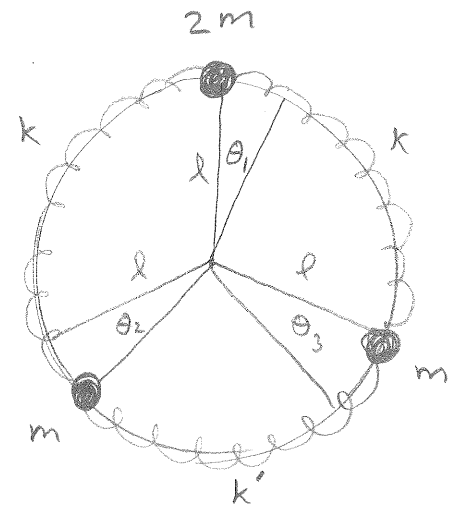
11.22 Fowles

a)



$$T = \frac{1}{2} 2ml^2\dot{\theta}_1^2 + \frac{1}{2} ml^2\dot{\theta}_2^2 + \frac{1}{2} ml^2\dot{\theta}_3^2 = (\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3)\underbrace{ml^2\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}}_{\overleftrightarrow{M}}\begin{pmatrix}\dot{\theta}_1\\\dot{\theta}_2\\\dot{\theta}_3\end{pmatrix}$$

$$V = \frac{1}{2} kl^2(\theta_1-\theta_3)^2 + \frac{1}{2} kl^2(\theta_1-\theta_2)^2 + \frac{1}{2}k'l^2(\theta_2-\theta_3)^2$$

$$= kl^2\theta_1^2 + \frac{1}{2}(k+k')l^2\theta_2^2 + \frac{1}{2}(k+k')l^2\theta_3^2$$
$$- kl^2\theta_1\theta_3 - kl^2\theta_1\theta_2 - k'l^2\theta_2\theta_3$$

$$V = \frac{1}{2}(\theta_1\ \theta_2\ \theta_3)\underbrace{l^2\begin{pmatrix}2k & -k & -k\\ -k & k+k' & -k'\\ -k & -k' & k+k'\end{pmatrix}}_{\overleftrightarrow{K}}\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix}$$

equations of motion

$$\overleftrightarrow{K}\vec{q} + \overleftrightarrow{M}\ddot{\vec{q}} = 0 \qquad \vec{q} \equiv \begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix}$$

Trial Solution

$$\vec{q} = \vec{a}\cos(\omega t - \delta)$$

plug Trial

$$\Rightarrow \overleftrightarrow{K}\vec{a} - \omega^2\overleftrightarrow{M}\vec{a} = 0 \Rightarrow$$

$$① \begin{pmatrix}2k-2\omega^2 m & -k & -k\\ -k & k+k'-\omega^2 m & -k'\\ -k & -k' & k+k'-\omega^2 m\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2k - 2\omega^2 m & -k & -k \\ -k & k+k'-\omega^2 m & -k' \\ -k & -k' & k+k'-\omega^2 m \end{vmatrix} = 0$$

make less messy by $\lambda \equiv \omega^2 m$

$$\Rightarrow (2k-2\lambda)(k+k'-\lambda)^2 - (2k-2\lambda)k'^2 + k(-k(k+k'-\lambda) - kk')$$
$$-k(kk' + k(k+k'-\lambda)) = 0$$

$$\Rightarrow \cancel{2}(k-\lambda)(k^2+k'^2+2kk'-2k\lambda-2k'\lambda+\lambda^2) - \cancel{2}(k-\lambda)k'^2 = 0$$
$$-\cancel{2}k^2(k'+k+k'-\lambda)$$

$$\Rightarrow k^3 + kk'^2 + 2k^2k' - 2k^2\lambda - 2kk'\lambda + k\lambda^2 - kk'^2 + k'^2\lambda$$
$$-k^2\lambda - k'^2\lambda - 2kk'\lambda + 2k\lambda^2 + 2k'\lambda^2 - \lambda^3 = 0$$
$$-k^3 - 2k^2k' + k^2\lambda$$

$$\Rightarrow kk'^2 - 2k^2\lambda - 4kk'\lambda - k'^2\lambda + 3k\lambda^2 + 2k'\lambda^2 - \lambda^3 = 0$$

$$\Rightarrow -(2k^2+4kk')\lambda + (3k+2k')\lambda^2 - \lambda^3 = 0$$

$$\Rightarrow \lambda\left[(2k^2+4kk') - (3k+2k')\lambda + \lambda^2\right] = 0$$

$$\Rightarrow \boxed{\omega_1^2 = 0} \qquad m\omega^2 = \frac{3k+2k' \pm \sqrt{9k^2+12kk'+4k'^2-8k^2-16kk'}}{2}$$

$$\Rightarrow m\omega^2 = \frac{3k+2k' \pm \sqrt{k^2-4kk'+4k'^2}}{2} = \frac{3k+2k'}{2} \pm \frac{(k-2k')}{2}$$

$$\Rightarrow \boxed{m\omega^2 = 2k,\ k+2k'}$$

<u>correct</u> same as I get by guessing

$$\Rightarrow \quad \omega_1 = 0 \qquad \omega_2^2 = \frac{2k}{m} \qquad \omega_3^2 = \frac{k+2k'}{m}$$

<u>mode 1</u>

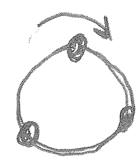
For $\omega_1 = 0$ rigid body mode

Plug into ① (the amplitude equation)

$$\Rightarrow k(2a_1 - a_2 - a_3) = 0 \quad \Rightarrow \quad \underline{2a_1 = a_2 + a_3} \quad \checkmark$$

$$-k a_1 + (k+k')a_2 - k' a_3 = 0$$

$$-k a_1 - k' a_2 \quad (k+k')a_3 = 0$$



like a rigid ring

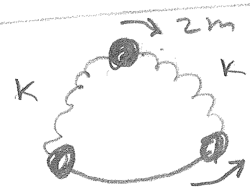
$a_{11} = a_{21} = a_{31}$ is a solution ✓

∴ the mode 1 is

$$\vec{Q}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (n_1 t + \delta_1)$$

where n_1 is a normalization constant.

<u>mode 2</u> $\quad \omega_2^2 = \frac{2k}{m}$



$\Leftarrow \quad \omega^2 = \frac{4k}{2m} = \frac{2k}{m}$

↑ k' spring not changing so it acts like its rigid 2m mass

Plug into ①

$$\Rightarrow \quad -2a_1 - a_2 - a_3 = 0$$

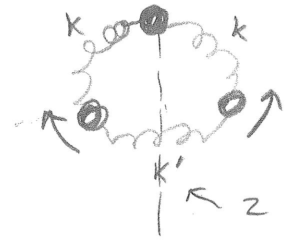
$$-k a_1 + (k'-k)a_2 - k' a_3 = 0$$

$$-k a_1 - k' a_2 + (k'-k)a_3 = 0$$

$a_{22} = a_{32} = -a_{12}$

$$\Rightarrow \quad \vec{Q}_2 = n_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cos(\omega_2 t - \delta_2)$$

mode 3 $\quad \omega_3^2 = \frac{k + 2k'}{m}$



2m mass not moving

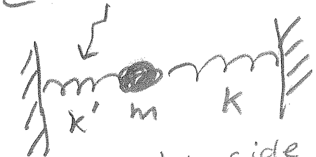
2 times the stretch

Plug into ①

$$\Rightarrow \quad 4k' a_1 - k a_2 - k a_3 = 0$$
$$-k a_1 - k' a_2 - k' a_3 = 0$$
$$-k a_1 - k' a_2 - k' a_3 = 0$$

like

cut in half doubles force k'



right side of circle

$$a_{23} = -a_{33} \qquad a_{13} = 0$$

$$\vec{Q}_3 = n_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos(\omega_3 t - \delta_3)$$

11.22 b) Particular Solution

starts at rest

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{pmatrix} = \vec{Q}_1 + \vec{Q}_2 + \vec{Q}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (A_1 t + \delta_1) + A_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cos(\omega_2 t - \delta_2)$$
$$+ A_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos(\omega_3 t - \delta_3)$$

$\theta_1(0) = \theta_{10}$ $\qquad \dot{\theta}_1(0) = 0$

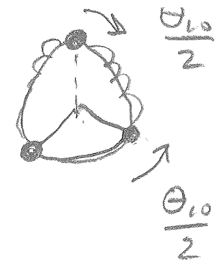
$\theta_2(0) = 0$ $\qquad \dot{\theta}_2(0) = 0$

$\theta_3(0) = 0$ $\qquad \dot{\theta}_3(0) = 0$

$$\Rightarrow \begin{pmatrix} \theta_{10} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \delta_1 + A_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cos\delta_2 + A_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos\delta_3$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + A_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \sin\delta_2 + A_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \sin\delta_3$$

6 equations 6 unknowns $A_1 \; A_2 \; A_3$
$\delta_1 \; \delta_2 \; \delta_3$



The intial conditions are a pure $\vec{Q}_2$ with a constant shift of $\frac{\theta_{10}}{2}$

$\delta_1 = \frac{\theta_{10}}{2}$ $\quad A_2 = -\frac{\theta_{10}}{2}$ $\quad$ all other constants are 0

$$\Rightarrow \boxed{\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{\theta_{10}}{2} - \frac{\theta_{10}}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cos \omega_2 t \qquad \theta_{10} = 10^\circ}$$

Guessing <u>Works</u>