This is taken from Thorton & Marion problem 14-3. The Book Version and the Verbose Version are the same.

Book Version

Show that the equation

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \tag{0.1}$$

a) is invariant under a Lorentz transformation but **b)** not under a Galilean transformation. (This is the wave equation that describes the propagation of light waves in free space.)

Verbose Version

Rewording Thorton's problem with more verbosity: Show that the equation

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \tag{0.2}$$

a) is invariant under a Lorentz transformation,

$$x_1' = \gamma\left(x_1 - vt\right), \tag{0.3}$$
$$x_2' = x_2, \tag{0.4}$$
$$x_3' = x_3, \tag{0.5}$$
$$t' = \gamma\left(t - \frac{vx_1}{c^2}\right), \tag{0.6}$$

where $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, v is the relative speed of the reference frame, and c is the speed of light.

b) Show that equation 0.2 is not invariant under a Galilean transformation

$$x_1' = x_1 - vt, \tag{0.7}$$
$$x_2' = x_2, \tag{0.8}$$
$$x_3' = x_3, \tag{0.9}$$
$$t' = t, \tag{0.10}$$

where v is the relative speed of the reference frame.

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x_1^2} + \frac{\partial^2\Psi}{\partial x_2^2} + \frac{\partial^2\Psi}{\partial x_3^2} \tag{0.11}$$

Getting Started

a) We need to show that the wave equation for $\Psi(x_1, x_2, x_3, t)$ when written in terms of x'_1, x'_2, x'_3, and t', is still in the form of the wave equation.

Start by defining $\Psi(x_1, x_2, x_3, t) \equiv \Psi'(x'_1, x'_2, x'_3, t')$. So we need to show that if

$$\frac{\partial^2\Psi}{\partial x_1^2} + \frac{\partial^2\Psi}{\partial x_2^2} + \frac{\partial^2\Psi}{\partial x_3^2} - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \tag{0.12}$$

than

$$\frac{\partial^2\Psi'}{\partial x_1'^2} + \frac{\partial^2\Psi'}{\partial x_2'^2} + \frac{\partial^2\Psi'}{\partial x_3'^2} - \frac{1}{c^2}\frac{\partial^2\Psi'}{\partial t'^2} = 0, \tag{0.13}$$

where x'_1, x'_2, x'_3, t', are related to x_1, x_2, x_3, t, but the Lorentz transformation, equations 0.3 through 0.6.

Remember Ψ' is the same as Ψ but with the variables transformed from the unprimed coordinates to the primed coordinates. Lets start with

$$\frac{\partial\Psi}{\partial x_1} = \frac{\partial x'_1}{\partial x_1}\frac{\partial\Psi'}{\partial x'_1} + \frac{\partial x'_2}{\partial x_1}\frac{\partial\Psi'}{\partial x'_2} + \frac{\partial x'_3}{\partial x_1}\frac{\partial\Psi'}{\partial x'_3} + \frac{\partial t'}{\partial x_1}\frac{\partial\Psi'}{\partial t'} \tag{0.14}$$

which from equations 0.3 through 0.6 simplifies to

$$\begin{aligned}
\frac{\partial\Psi}{\partial x_1} &= \gamma\frac{\partial\Psi'}{\partial x'_1} + 0\frac{\partial\Psi'}{\partial x'_2} + 0\frac{\partial\Psi'}{\partial x'_3} + \gamma\left(\frac{-v}{c^2}\right)\frac{\partial\Psi'}{\partial t'} && (0.15)\\
&= \gamma\frac{\partial\Psi'}{\partial x'_1} + \gamma\left(\frac{-v}{c^2}\right)\frac{\partial\Psi'}{\partial t'}, && (0.16)
\end{aligned}$$

and so

$$\begin{aligned}
\frac{\partial^2\Psi}{\partial x_1^2} &= \frac{\partial}{\partial x_1}\frac{\partial\Psi}{\partial x_1} = \frac{\partial}{\partial x_1}\left[\gamma\frac{\partial\Psi'}{\partial x'_1} + \gamma\left(\frac{-v}{c^2}\right)\frac{\partial\Psi'}{\partial t'}\right] && (0.17)\\
&= \gamma\frac{\partial}{\partial x'_1}\left[\gamma\frac{\partial\Psi'}{\partial x'_1} + \gamma\left(\frac{-v}{c^2}\right)\frac{\partial\Psi'}{\partial t'}\right] + \gamma\left(\frac{-v}{c^2}\right)\frac{\partial}{\partial t'}\left[\gamma\frac{\partial\Psi'}{\partial x'_1} + \gamma\left(\frac{-v}{c^2}\right)\frac{\partial\Psi'}{\partial t'}\right] && (0.18)\\
&= \gamma^2\frac{\partial^2\Psi'}{\partial x_1'^2} - \gamma^2\frac{v}{c^2}\frac{\partial^2\Psi'}{\partial x'_1\partial t'} - \gamma^2\frac{v}{c^2}\frac{\partial^2\Psi'}{\partial t'\partial x'_1} + \left(\frac{\gamma v}{c^2}\right)^2\frac{\partial^2\Psi'}{\partial t'^2}; && (0.19)
\end{aligned}$$

and so that's one term down. Note that v, γ, and c are constants. Find $\frac{\partial^2\Psi}{\partial x_2^2}$, $\frac{\partial^2\Psi}{\partial x_3^2}$, and $\frac{\partial^2\Psi}{\partial t^2}$ in a simular fashion and than plug them all into equation 0.12 and that should reduce to equation 0.13.

b) We need to show that the wave equation for $\Psi(x_1, x_2, x_3, t)$ when written in terms of x'_1, x'_2, x'_3, and t', is not still in the form of the wave equation when using the Galilean transformation, equations 0.7 through 0.10.