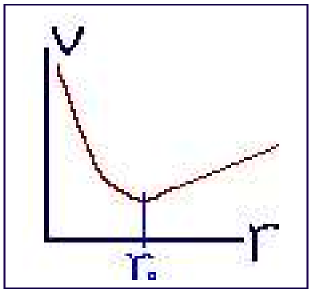
A particle of mass m moves in one dimension, r, with the following potential energy

$$V(r) = ar + \frac{b}{r^2}, \tag{0.1}$$

where a and b are positive constants, and the position of the particle, r, is always positive.

1 Plot $V(r)$

Make a rough plot of $V(r)$, for $r > 0$.



2 Force

Find the force, $F(r)$, from this potential as a function of r.

$$F(r) = -\frac{\partial V(r)}{\partial r} \qquad \Rightarrow \qquad F(r) = -a + \frac{2b}{r^3} \tag{2.1}$$

3 Equilibrium Position

Find r_0, the one equilibrium r position of the particle and mark this on your plot.

$$r_0 = \sqrt[3]{\frac{2b}{a}} \tag{3.1}$$

4 Expanding about the Equilibrium Position

When the particle is displaced a small amount from r_0 in the positive r direction or the negative r direction it is pushed back to $r = r_0$ by the force from this potential. We call this equilibrium position, $r = r_0$, a stable equilibrium position. The shape of $V(r)$ at, or near, $r = r_0$, is concave up, like a valley.

An unstable equilibrium position is like the top of a hill, if you move a small amount to either side of the top you will fall down the hill away from the top.

Expand $V(r)$ as a Taylor series about $r = r_0$ up to the $(r - r_0)^2$ term. Recall that a Taylor series expansion has the form

$$V(r) \approx \sum_{n=0}^{N} \frac{(r - r_0)^n}{n!} \frac{\mathrm{d}^n V(r_\star)}{\mathrm{d}r_\star{}^n}\bigg|_{r_\star = r_0}. \tag{4.1}$$

$$V(r) \approx V(r_0) + V'(r_0)\,[r - r_0] + \frac{1}{2}V''(r_0)\,[r - r_0]^2 = ar_0 + \frac{b}{r_0^2} + \frac{1}{2}\frac{6b}{r_0^4}\,(r - r_0)^2 \tag{4.2}$$

5 Small Oscillations about the Equilibrium Position

Note that when $V(r)$ is expanded about $r = r_0$ it has the same form as the potential for a 1-D simple harmonic oscillator

$$V(x) = V_0 + \frac{1}{2}kx^2, \tag{5.1}$$

where $x = r - r_0$, and V_0 is a constant, and k is the spring constant.

5.1

For our potential, $V(r)$, what is the spring constant, k, when we are near $r = r_0$? Answer in terms of constants a and b.

$$k = \frac{3a^{\frac{4}{3}}}{\sqrt[3]{2b}} \tag{5.2}$$

5.2

What will be the angular frequency of oscillation, ω_0, of the particle about the equilibrium position $r = r_0$?

$$\omega_0 = \frac{\sqrt{3}a^{\frac{2}{3}}}{\sqrt[6]{2b}\sqrt{m}} \tag{5.3}$$

6 Another Way

6.1 Equation of Motion

Write the equation of motion of the particle.

$$F(r) = -a + \frac{2b}{r^3} \qquad \Rightarrow \qquad m\ddot{r} = -a + \frac{2b}{r^3} \tag{6.1}$$

6.2 Expand the Equation of Motion about the Equilibrium Position

Expand the equation of motion about the equilibrium position, r_0, by making the substitution $r \to r_0 + \epsilon$, where ϵ is small compared to r_0, and show that the equation of motion of ϵ is that of simple harmonic motion. Recall the series expansion $(1+x)^n \approx 1 + nx$ for small x.

$$r = r_0 + \epsilon \qquad \Rightarrow \qquad \ddot{r} = \ddot{\epsilon} \tag{6.2}$$

$$\frac{1}{r^3} = \frac{1}{(r_0+\epsilon)^3} = \frac{1}{{r_0}^3\left(1+\frac{\epsilon}{r_0}\right)^3} = \frac{1}{{r_0}^3}\left(1+\frac{\epsilon}{r_0}\right)^{-3} \approx \frac{1}{{r_0}^3}\left(1-3\frac{\epsilon}{r_0}\right) \tag{6.3}$$

Combining this with the equation of motion gives

$$m\ddot{\epsilon} = -a + \frac{2b}{{r_0}^3}\left(1-3\frac{\epsilon}{r_0}\right) \qquad \Rightarrow \qquad m\,\ddot{\epsilon} = -a + \frac{2b}{{r_0}^3} - \frac{6b}{{r_0}^4}\epsilon \qquad \Rightarrow \qquad m\,\ddot{\epsilon} = -\frac{6b}{{r_0}^4}\epsilon \tag{6.4}$$

$$\Rightarrow \qquad m\,\ddot{\epsilon} = -\frac{3a^{\frac{4}{3}}}{\sqrt[3]{2b}}\epsilon \qquad \Rightarrow \qquad k = \frac{3a^{\frac{4}{3}}}{\sqrt[3]{2b}} \qquad \text{and} \qquad \omega_0 = \frac{\sqrt{3}a^{\frac{2}{3}}}{\sqrt[6]{2b}\sqrt{m}} \tag{6.5}$$

as before.