A particle of mass m moves in one dimension, r, with the following potential energy

$$V(r) = ar + \frac{b}{r^2}, \tag{0.1}$$

where a and b are positive constants, and the position of the particle, r, is always positive.

1 Plot $V(r)$

Make a rough plot of $V(r)$, for $r > 0$.

2 Force

Find the force, $F(r)$, from this potential as a function of r.

3 Equilibrium Position

Find r_0, the one equilibrium r position of the particle and mark this on your plot.

4 Expanding about the Equilibrium Position

When the particle is displaced a small amount from r_0 in the positive r direction or the negative r direction it is pushed back to $r = r_0$ by the force from this potential. We call this equilibrium position, $r = r_0$, a stable equilibrium position. The shape of $V(r)$ at, or near, $r = r_0$, is concave up, like a valley.

An unstable equilibrium position is like the top of a hill, if you move a small amount to either side of the top you will fall down the hill away from the top.

Expand $V(r)$ as a Taylor series about $r = r_0$ up to the $(r - r_0)^2$ term. Recall that a Taylor series expansion has the form

$$V(r) \approx \sum_{n=0}^{N} \frac{(r - r_0)^n}{n!} \left. \frac{\mathrm{d}^n V(r_\star)}{\mathrm{d}{r_\star}^n} \right|_{r_\star = r_0} . \tag{4.1}$$

5 Small Oscillations about the Equilibrium Position

Note that when $V(r)$ is expanded about $r = r_0$ it has the same form as the potential for a 1-D simple harmonic oscillator

$$V(x) = V_0 + \frac{1}{2}kx^2, \tag{5.1}$$

where $x = r - r_0$, and V_0 is a constant, and k is the spring constant.

5.1

For our potential, $V(r)$, what is the spring constant, k, when we are near $r = r_0$? Answer in terms of constants a and b.

5.2

What will be the angular frequency of oscillation, ω_0, of the particle about the equilibrium position $r = r_0$?

6 Another Way

6.1 Equation of Motion

Write the equation of motion of the particle.

6.2 Expand the Equation of Motion about the Equilibrium Position

Expand the equation of motion about the equilibrium position, r_0, by making the substitution $r \to r_0 + \epsilon$, where ϵ is small compared to r_0, and show that the equation of motion of ϵ is that of simple harmonic motion. Recall the series expansion $(1+x)^n \approx 1 + nx$ for small x.