HW 5 Waves 5

1.1)

$$\vec{Q}_N = \begin{pmatrix} a_{1N} \\ a_{2N} \\ a_{3N} \end{pmatrix} \cos(\omega_N t - \delta_N) \qquad \text{mode } N$$

Fowles 11.5.18 $\quad a_{kN} = \sin\left(\frac{N\pi k}{n+1}\right)$

k-th particle
N-th mode
n particles

Fowles 11.5.17 $\quad \omega_N = 2\omega_0 \sin\left(\frac{N\pi}{2n+2}\right) \qquad \omega_0^2 = \frac{K}{m} = \frac{T}{L/4}\left(\frac{1}{m/3}\right) = \frac{12T}{mL}$

$n = 3 \quad N = 1, 2, 3 \quad k = 1, 2, 3$

$N = 1 \quad a_{k1} = \sin\left(k\frac{\pi}{4}\right) \qquad \omega_1 = 2(2)\sqrt{3}\left(\frac{T}{mL}\right)^{1/2} \sin\left(\frac{\pi}{8}\right)$

$$= 4\sqrt{3}\sqrt{\frac{T}{mL}} \sin\left(\frac{\pi}{8}\right)$$

$$\boxed{\vec{Q}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cos(\omega_1 t - \delta_1), \quad \omega_1 = 4\sqrt{3}\sqrt{\frac{T}{mL}} \sin\frac{\pi}{8} \simeq 2.65\sqrt{\frac{T}{mL}}}$$

$N = 2 \quad a_{k2} = \sin\left(\frac{2\pi k}{4}\right) \qquad \omega_2 = 4\sqrt{3}\sqrt{\frac{T}{mL}} \sin\left(\frac{\pi}{4}\right) = 4\sqrt{\frac{3}{2}\frac{T}{mL}}$

$$\boxed{\vec{Q}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cos(\omega_2 t - \delta_2), \quad \omega_2 = 4\sqrt{\frac{3T}{2mL}} \simeq 4.90\sqrt{\frac{T}{mL}}}$$

$a_{k3} = \sin\left(\frac{3\pi}{4}k\right) \qquad \omega_3 = 4\sqrt{3}\left(\frac{T}{mL}\right)^{1/2} \sin\left(\frac{3\pi}{8}\right)$

$$\boxed{\vec{Q}_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cos(\omega_3 t - \delta_3) \,, \quad \omega_3 = 4\sqrt{3}\sqrt{\frac{T}{mL}} \sin\left(\frac{3\pi}{8}\right) \simeq 6.40\sqrt{\frac{T}{mL}}}$$

1.2) By inspection $\delta_1 = \delta_2 = \delta_3 = 0 \qquad A_2 = 0$

$$\vec{q}(t=0) = \begin{pmatrix} \frac{A}{2} \\ A \\ \frac{A}{2} \end{pmatrix} = A_1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} + A_3 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\dot{\vec{q}}(t=0) = 0 = 0 + 0 \quad \checkmark$$

$$\Rightarrow \frac{A}{2} = \frac{A_1}{\sqrt{2}} + \frac{A_3}{\sqrt{2}} \,, \quad A = A_1 - A_3 \Rightarrow A_1 = A + A_3$$

$$\Rightarrow A_3 = -\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right) A \qquad A_1 = \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) A$$

$$\Rightarrow \boxed{\vec{q}(t) = \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) A \begin{pmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{pmatrix} \cos \omega_1 t - \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right) A \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cos \omega_3 t}$$

2.1) $v^2 = \dfrac{T}{\left(\frac{m}{L}\right)} \Rightarrow \boxed{v = \sqrt{\dfrac{TL}{m}}}$

2.2 $\omega = 2\pi \dfrac{v}{\lambda} \,, \quad \lambda = \dfrac{2L}{N} \Rightarrow \omega_N = \pi \dfrac{v}{L} N = \pi \sqrt{\dfrac{T}{mL}}\, N$

$$\Rightarrow \boxed{\omega_1 = \pi\sqrt{\frac{T}{mL}} \,, \quad \omega_2 = 2\pi\sqrt{\frac{T}{mL}} \,, \quad \omega_3 = 3\pi\sqrt{\frac{T}{mL}}}$$

2.3)

$$\frac{\omega_{N\,part}^2}{\omega_{N\,string}^2} = \frac{2^2 \sin^2\left(\frac{N\pi}{8}\right) \frac{12T}{mL}}{\pi^2 \frac{T}{mL} N^2}$$

$$\Rightarrow \frac{\omega_{N\,part}}{\omega_{N\,string}} = \frac{2\left[\sin\left(\frac{N\pi}{8}\right)\right]2\sqrt{3}}{\pi N} = \frac{4\sqrt{3}}{\pi} \frac{\sin\left(\frac{N\pi}{8}\right)}{N}$$

$$\boxed{\begin{aligned} \frac{\omega_{1\,part}}{\omega_{1\,string}} &= \frac{4\sqrt{3}}{\pi} \sin\left(\frac{\pi}{8}\right) \simeq 0.844 \\ \frac{\omega_{2\,part}}{\omega_{2\,string}} &= \frac{2\sqrt{3}}{\pi} \sin\left(\frac{\pi}{4}\right) \simeq 0.780 \\ \frac{\omega_{3\,part}}{\omega_{3\,string}} &= \frac{4}{\pi\sqrt{3}} \sin\left(\frac{3\pi}{8}\right) \simeq 0.679 \end{aligned}}$$

2.4)

$$\lambda_N = \frac{2L}{N} \Rightarrow \frac{2\pi x}{\lambda_N} = \frac{\pi x}{L} N$$

Plugging (2.1) → (2.3) at $t = 0$

$$\Rightarrow A_M = \frac{2}{L} \int_{x=0}^{L} \sum_{N=1}^{\infty} A_N \sin\left(\frac{N\pi x}{L}\right) \sin\left(\frac{M\pi x}{L}\right) dx$$

$$= \frac{2}{L} \sum_{N=1}^{\infty} A_N \int_{x=0}^{L} \sin\left(\frac{N\pi x}{L}\right) \sin\left(\frac{M\pi x}{L}\right) dx$$

change of variables $y = \frac{\pi x}{L}$ $\quad dy = \pi \frac{dx}{L}$ $\quad x = 0 \; y = 0$, $\; x = L \; y = \pi$

$$\Rightarrow A_M = 2\sum_{N=1}^{\infty} A_N \frac{1}{\pi}\int_0^{\pi} \sin NY \sin MY\, dy = \frac{2}{\pi}\sum_{N=1}^{\infty} A_N \left(\frac{\pi}{2}\delta_{N,M}\right)$$

$$= A_M \qquad \text{QED}$$

2.5)
$$A_M = \frac{2}{L}\left[\int_0^{\frac{L}{2}} \frac{2A}{L} x \sin \frac{\pi M}{L} x\, dx + \int_{L/2}^{L} \left(2A - \frac{2A}{L}x\right)\sin\left(\frac{\pi M x}{L}\right) dx\right]$$

change of variables $y = \frac{M\pi}{L}x \quad dy = \frac{M\pi}{L}x$

$x = 0 \quad y = 0$
$x = \frac{L}{2} \quad y = \frac{M\pi}{2}$
$x = L \quad y = M\pi$

$$\Rightarrow A_M = \frac{2}{L}\Bigg[\frac{2A}{L}\left(\frac{L}{M\pi}\right)^2 \int_0^{\frac{M\pi}{2}} y \sin y\, dy + 2A\left(\frac{L}{M\pi}\right)\int_{\frac{M\pi}{2}}^{M\pi} \sin y\, dy$$

$$- \left(\frac{2A}{L}\right)\left(\frac{L}{M\pi}\right)^2 \int_{\frac{M\pi}{2}}^{M\pi} y \sin y\, dy \Bigg]$$

$$= \frac{4A}{M^2\pi^2}\left(\sin y - y\cos y\right)\Big|_0^{\frac{M\pi}{2}} + \frac{4A}{M\pi}\left(-\cos y\right)\Big|_{\frac{M\pi}{2}}^{M\pi}$$

$$- \frac{4A}{M^2\pi^2}\left(\sin y - y\cos y\right)\Big|_{\frac{M\pi}{2}}^{M\pi}$$

$$= \frac{4A}{M^2\pi^2}\sin\frac{M\pi}{2} - \cancel{\frac{4A}{M^2\pi^2}\frac{M\pi}{2}\cos\left(\frac{M\pi}{2}\right)} + \cancel{\frac{4A}{M\pi}\cos\frac{M\pi}{2}} - \cancel{\frac{4A}{M\pi}\cos M\pi}$$

$$- \frac{4A}{M^2\pi^2}\cancelto{0}{\sin M\pi} + \frac{4A}{M^2\pi^2}\sin\frac{M\pi}{2} + \cancel{\frac{4A}{M^2\pi^2}M\pi\cos M\pi} - \cancel{\frac{4A}{M^2\pi^2}\frac{M\pi}{2}\cos\frac{M\pi}{2}}$$

$$= \frac{8A}{M^2\pi^2}\sin\frac{M\pi}{2} = \frac{8A}{M^2\pi^2}\begin{cases} 0 & M \text{ even} \\ (-1)^{(M-1)/2} & M \text{ odd} \end{cases}$$

Flips sign every other

$$\Rightarrow \boxed{\begin{aligned} Y(x,t) &= \sum_{M=1,3,5\cdots}^{\infty} \frac{8A}{M^2\pi^2}(-1)^{\frac{(M-1)}{2}} \sin\left(\frac{M\pi x}{L}\right)\cos(M\omega_1 t) \\ \omega_1 &= \pi\sqrt{\frac{T}{mL}} \end{aligned}}$$

3.1)

$$V = \frac{1}{2}k\left[\left(l^2 + (Y_n - Y_{n-1})^2\right)^{1/2} - l\right]^2$$

$$+ \frac{1}{2}k\left[\left(l^2 + (Y_{n+1} - Y_n)^2\right)^{1/2} - l\right]^2$$

$$= \frac{1}{2}k l^2\left[\left(1 + \left(\frac{Y_n}{l} - \frac{Y_{n-1}}{l}\right)^2\right)^{1/2} - 1\right]^2$$

$$+ \frac{1}{2}k l^2\left[\left(1 + \left(\frac{Y_{n+1}}{l} - \frac{Y_n}{l}\right)^2\right)^{1/2} - 1\right]^2$$

$\frac{Y_{n+1}}{l}$ and $\frac{Y_n}{l}$ and $\frac{Y_{n-1}}{l} << 1$ $\qquad (1+\epsilon)^{1/2} \simeq 1 + \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2)$

$$\Rightarrow V \simeq \frac{1}{2}k l^2\left[\cancel{1} + \frac{1}{2}\left(\frac{Y_n}{l} - \frac{Y_{n-1}}{l}\right)^2 - \cancel{1}\right]^2$$

$$+ \frac{1}{2}k l^2\left[\cancel{1} + \frac{1}{2}\left(\frac{Y_{n+1}}{l} - \frac{Y_n}{l}\right)^2 - \cancel{1}\right]^2$$

$$\simeq \frac{1}{2}k l^2 \frac{1}{4}\left(\frac{Y_n}{l} - \frac{Y_{n-1}}{l}\right)^4 + \frac{1}{2}k l^2 \frac{1}{4}\left(\frac{Y_{n+1}}{l} - \frac{Y_n}{l}\right)^4$$

$$\simeq \frac{1}{2}\left(\frac{k}{l^2}\right)\frac{1}{4}(Y_n - Y_{n-1})^4 + \frac{1}{2}\left(\frac{k}{l^2}\right)\frac{1}{4}(Y_{n+1} - Y_n)^4$$

Goes like the 4-th power of $Y$; Not simple Harmonic

3.2

$$V = F\left[\left(\ell^2 + (y_n - y_{n-1})^2\right)^{1/2} - \ell\right]$$
$$+ F\left[\left(\ell^2 + (y_{n+1} - y_n)^2\right)^{1/2} - \ell\right]$$

$$\Rightarrow V = F\ell\left[\left(1 + \left(\frac{y_n}{\ell} - \frac{y_{n-1}}{\ell}\right)^2\right)^{1/2} - 1\right]$$
$$+ F\ell\left[\left(1 + \left(\frac{y_{n+1}}{\ell} - \frac{y_n}{\ell}\right)^2\right)^{1/2} - 1\right]$$

$$V \simeq F\ell\left[1 + \frac{1}{2}\left(\frac{y_n}{\ell} - \frac{y_{n-1}}{\ell}\right)^2 - 1\right]$$
$$+ F\ell\left[1 + \frac{1}{2}\left(\frac{y_{n+1}}{\ell} - \frac{y_n}{\ell}\right)^2 - 1\right]$$

$$= \frac{1}{2}\frac{F}{\ell}\left(y_n - y_{n-1}\right)^2 + \frac{1}{2}\frac{F}{\ell}\left(y_{n+1} - y_n\right)^2$$

Goes like the square of $y_i$

$\Rightarrow$ Simple Harmonic Motion

3.3

The case with springs does not have simple harmonic Motion.

The case with a string has simple harmonic motion.