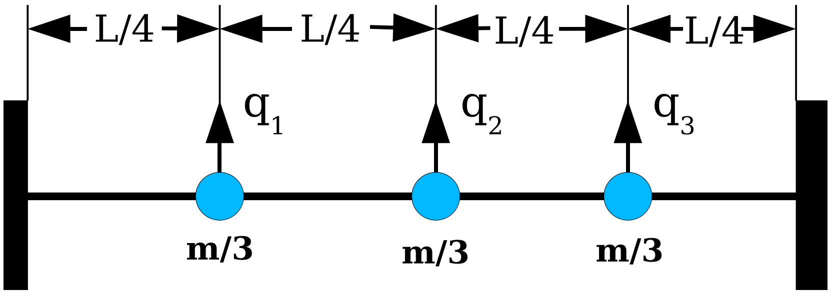
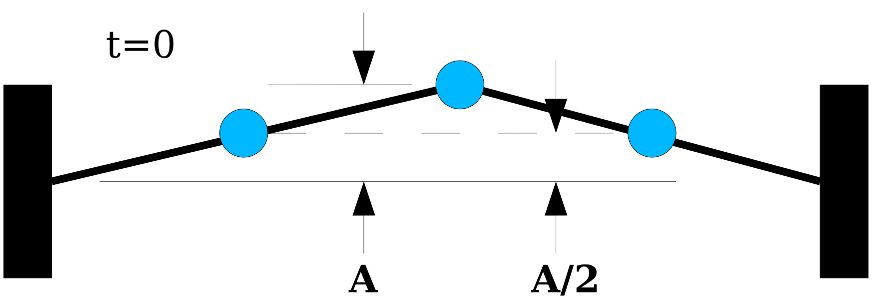
1 Particles on a Massless String

1.1 Finding the Normal Modes



The figure above shows three particles, of mass $\frac{m}{3}$ each, attached to a string that keeps a constant tension, T, in it. The length of the whole string is L. The string does not move up and down at the two ends. The tension remains constant even then the particles move a little. We will consider the particles to move a small distance in the up and down (transverse) direction. Find the three normal modes and list them in the order $N = 1, 2, 3$, where a coordinate position vector, $\overrightarrow{q}$, is defined by $\overrightarrow{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$. Write them in the form $\overrightarrow{Q_N} = \begin{pmatrix} a_{1N} \\ a_{2N} \\ a_{3N} \end{pmatrix} cos(\omega_N t - \delta_N)$, where $N = 1, 2, 3$. Write the normal angular frequencies, ω_N, $N = 1, 2, 3$ in terms of T, m, and L. Normalizing the amplitudes of the modes is not required. You should use Fowles equations 11.5.14d to 11.5.20 (pages 502-503) to solve this problem. The pictures on page 503 (figure 11.5.2) goes a long way in helping understand these modes.

1.2 Find a Particular Solution



The middle particle is pulled up (plucked) a distance of A and released, at time $t = 0$, as shown. When the particle is released none of the particles are initially moving. Find an expression for the motion of the three particles for all time, t, after they have been released. Write your solution as a linear combination of the normal mode vectors, $\overrightarrow{Q_N}$, where $N = 1, 2, 3$ which you found above, including the phase constants δ_N, where $N = 1, 2, 3$. So your solution will be of the form

$$\overrightarrow{q} = A_1\overrightarrow{Q_1} + A_2\overrightarrow{Q_2} + A_3\overrightarrow{Q_3} = A_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} cos(\omega_1 t - \delta_1) + A_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} cos(\omega_2 t - \delta_2) + A_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} cos(\omega_3 t - \delta_3) \quad (1.1)$$

More specifically, find the values for the itegration constants A_1, A_2, A_3, δ_1, δ_2, and δ_3, and present them in the form of equation 1.1. Looking at the pictures on Fowles page 503 (figure 11.5.2) helps a lot here. You can eliminate many unknowns from symmetry considerations.

2 Compare to a String

2.1 Speed of a Wave on the String

A string is stretched to a length L. It has a total mass of m. The tension in the string is kept constant with a value of T. This is some what similar to the above three particle problem.

Find the speed of a wave on this string. Fowles equation 11.6.7b may be handy. Answer in terms of L, m, and T.

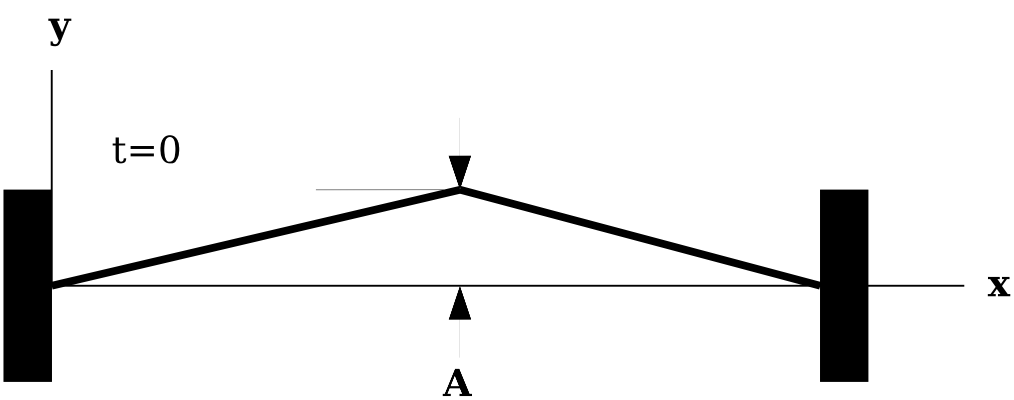
2.2 Normal Angular Frequencies

Find the angular frequency of the first three lowest frequency modes. Give your answer in terms of m, L, and T. Fowles equations 11.6.12 and 11.6.16 may be handy.

2.3 Compare with Three Particle Problem

For the first lowest three angular frequency modes, find the ratio of the three particle frequency to the spring angular frequency. That's something like $\frac{\omega_{1particles}}{\omega_{1string}}$, $\frac{\omega_{2particles}}{\omega_{2string}}$, and $\frac{\omega_{3particles}}{\omega_{3string}}$. The answer to this should be just three constant numbers. Give the exact numbers and than three significant figures.

2.4 A Some What General Solution



The string is plucked at the center, at $x = \frac{L}{2}$, and released from a rest position, as shown. Now we'll find an expression for the shape of the string as a function of time. From Fowles equation 11.6.14 the answer can be put in the form

$$y(x,t) = \sum_{N=1}^{\infty} A_N sin\left(\frac{2\pi x}{\lambda_N}\right) cos(\omega_N t), \tag{2.1}$$

where the sum is over all modes, and $\lambda_N = \frac{2L}{N}$ and ω_N are related by $\frac{\omega_N}{2\pi} = \frac{v}{\lambda_N}$, which is Fowles equation 11.6.12.

In this case the mode coefficients, A_N, can be found from the initial conditions $\dot{y}(x,0) = 0$ and

$$y(x,0) = \begin{cases} \frac{2A}{L}x & : \quad 0 \le x \le \frac{L}{2} \\ 2A - \frac{2A}{L}x & : \quad \frac{L}{2} \le x \le L \end{cases} \tag{2.2}$$

and from Fourier series

$$A_M = \frac{2}{L}\int_{x=0}^{L} y(x,0)\sin\left(\frac{2\pi}{\lambda_M}x\right)dx \tag{2.3}$$

where M is a particular integer in the sum in equation 2.1, and λ_M is the wave length of the M-th mode (Fowles equation 11.6.16 with $M \to N$).

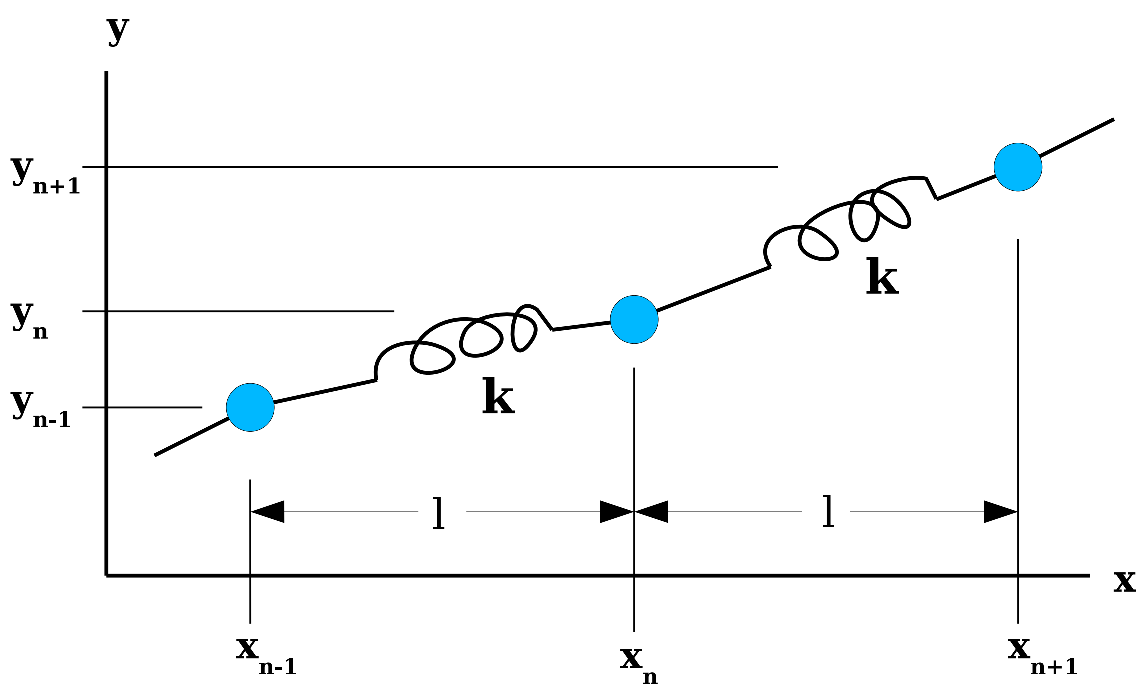
Note: The initial condition $\dot{y}(x,0) = 0$ is already satisfied, so we have not started with the most general solution to the 1-D wave equation of this system.

Show that equation 2.3 is the expression for the mode (Fourier) coefficients for any $y(x,0)$ (not just $y(x,0)$ in equation 2.2) given that the string starts at rest, and $y(0,t) = 0$ and $y(L,t) = 0$, by plugging $y(x,0)$, from equation 2.1, with $t = 0$, into equation 2.3.

2.5 Find a Particular Solution

Plug equation 2.2 into equation 2.3 to find the mode (Fourier) coefficients, A_N, for equation 2.1 to get the shape of the string as a function of time. Put your answer in terms of the lowest angular frequency ω_1 and L.

3 Why Don't We Study Transverse Motions of Particles on a Line of Springs?



3.1 Spring Potential Energy

Above is shown a section of a linear array of masses (particles) and springs. Both springs shown above have spring constants k. The masses have an equilibrium position with x positions at even l intervals and $y_i = 0$ for all masses. Assume that $y_{n-1} \ll l$, $y_n \ll l$, and $y_{n+1} \ll l$. Calculate the potential energy in the two springs and expand this expression to the lowest power in $\frac{y_{n-1}}{l}$, $\frac{y_n}{l}$, and $\frac{y_{n+1}}{l}$.

3.2 String Potential Energy

Calculate and expand potential energy again, but now replacing the springs with string that keeps a constant force, F, for all values of string elongation. The potential energy change in the stretching string will be F times the change in the length of the string. Expand the potential energy in the two strings to the lowest power in $\frac{y_{n-1}}{l}$, $\frac{y_n}{l}$, and $\frac{y_{n+1}}{l}$.

3.3 Why?

Compare the forms of the spring potential energy and the string potential energy with the forms that we have been studying, and answer the question: Why don't we study transverse motions of particles on a line of springs?